Explicit Isogenies of prime degree over number fields

work in progress with Maarten Derickx

Barinder Singh Banwait

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Atelier PARI/GP
Université de Franche-Comté, Besançon
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UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

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> *Ces motifs ne permettront pas de se déplacer pour raisons touristiques ou professionnelles*

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Test date: 06/01/2022

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Self-isolate immediately (including if this is a follow-up test result) from the day your symptoms started, or the test date if you've no symptoms.

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Fortunately it has been mostly mild.

Introduction

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- The degree of ϕ $= |\ker(\phi)| = [\overline{K}(E_1) : \phi^*\overline{K}(E_2)]$.
- ϕ is K-rational if it is compatible with the G_K-action on E_1 and E_2; that is, if the following diagram commutes for all $\sigma \in G_K$:

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- ϕ is said to be cyclic if $\ker(\phi)$ is a cyclic group.

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Fact

Every isogeny is the composition of a cyclic isogeny with the multiplication-by-m map for some $m \geq 1$.

So between any two elliptic curves in the isogeny class of E, there is a unique minimal cyclic isogeny degree between them.

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The degree computation is based on Billerey's algorithm for computing isogenies of prime degree for a fixed elliptic curve E/K.

Nicolas Billerey

Uniform isogeny primes?

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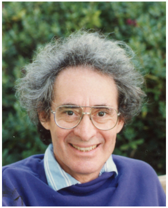
Theorem (Mazur, 1978)

$$\mathrm{IsogPrimeDeg}(\mathbb{Q}) = \{2, 3, 5, 7, 11, 13, 17, 19, 37, 43, 67, 163\}$$

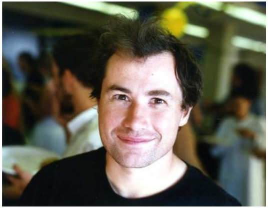


Barry C. Mazur

Theorem (Momose + Merel, 1995)

Assume GRH. Then $\mathrm{IsogPrimeDeg}(K)$ *is finite if and only if* K *does not contain the Hilbert class field of an imaginary quadratic field.*

Fumiyuki Momose



Löic Merel

Isogeny Prime v1

Computing IsogPrimeDeg(K)?

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Theorem (B.-Derickx)

Let K be a number field which does not contain the Hilbert class field of an imaginary quadratic field. Then there is an algorithm which computes a superset of $\mathrm{IsogPrimeDeg}(K)$ *as the union of three sets:*

$$\mathrm{IsogPrimeDeg}(K) \subseteq \mathrm{PreTypeOneTwoPrimes}(K) \cup \mathrm{TypeOnePrimes}(K) \cup \mathrm{TypeTwoPrimes}(K).$$

With Maarten Derickx in West London last week

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Theorem (Momose, watered-down)

Let K be a number field which does not contain the HCF of an IQF. Then there exists a constant $C_0 = C_0(K)$ such that for any prime $p > C_0$, and for any elliptic curve admitting a K-rational p-isogeny, the isogeny character λ falls into one of the following two types:

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LEMMA 1. *Assume that k is a Galois extension of $\mathbf{Q}$ and that the rational prime p is unramified in k. Then for a fixed prime $\mathfrak{p}$ of k lying over p, we have integers a_σ, $0 \leqslant a_\sigma \leqslant 12$, for $\sigma \in \mathrm{Gal}(k/\mathbf{Q})$ such that*

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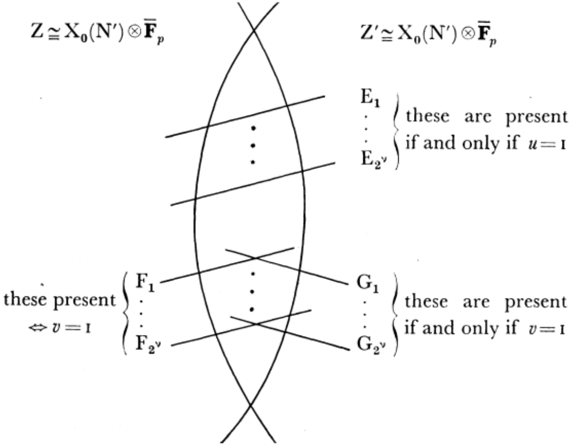
REMARK 1. The integers $a_\mathfrak{P}$'s take the values $0, 12; 4, 8$ (only if the modular invariant $j(E) \equiv 0 \pmod{\mathfrak{p}}$ and $p \equiv 2 \pmod 3$); 6 (only if $j(E) \equiv 1728 \pmod{\mathfrak{p}}$ and $p \equiv 3 \pmod 4$) (cf. [Ma1], Chap. 3; [Ma2]).

We obtain the following picture for the reduction modulo p of $X_0(N)$:



From Mazur and Rapoport's Appendix to Mazur's 1977 paper. The E components arise from $j = 1728$ elliptic curves, the F and G from $j = 0$.

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For the other signatures ε one can construct a non-zero integer $ABC(\varepsilon, \mathfrak{q})$ (for prime ideals $\mathfrak{q}$ of K) which multiplicatively bounds the isogeny primes with that signature.

TypeOnePrimes

Let E/K be an elliptic curve with a K-rational p-isogeny of Type 1. Replacing this isogeny with its dual if necessary, we may suppose that $\lambda^{12h_K} = 1$, i.e., $\epsilon = (0, \cdots, 0)$.

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i.e. we can multiplicatively bound this case.

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In the first case: if any of the embedded points x^σ specializes to $0_{/\mathbb{F}_\mathfrak{q}}$, then we again get a non-zero multiplicative bound.

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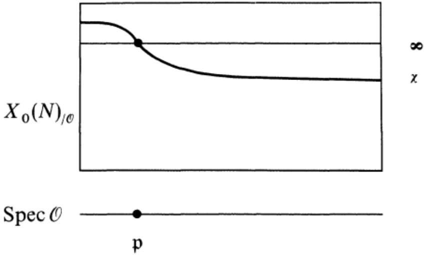
By an analogue of **Mazur's specialization lemma**, we obtain that $f(x^\sigma) = f(\infty,\cdots,\infty)$.

We have, therefore, the following state of affairs: the two $\mathcal{O}$-sections of $X_0(N)$, $x_{/\mathcal{O}}$ and $\infty_{/\mathcal{O}}$, "cross" at $\mathfrak{p}$, and map to the same section of A under $f_{/\mathcal{O}}$ (the zero-section). But this contradicts the fact that f is a formal immersion at $\infty_{/k(\mathfrak{p})}$.



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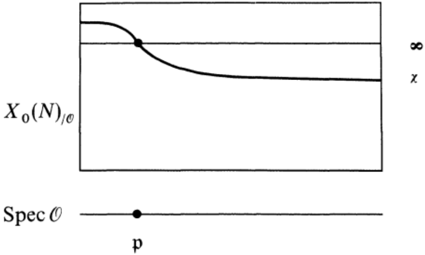

We conclude that f is *not* a formal immersion at $(\infty, \cdots, \infty)$.

The set of such ps is very small and can be explicitly bounded.

Proposition 5.3. *Let* $H \subseteq (\mathbb{Z}/p\mathbb{Z})^\times/\{\pm 1\}$ *be a subgroup. Let* $\ell \neq p$ *be a prime and consider* $t = t_1(t_0)$ *as in Proposition 5.1 when* ℓ *is odd, or* t *as in Corollary 5.2 when* $\ell = 2$. *Then* $t \circ \iota$ *is a formal immersion at all* $\bar{x}_H \in X_H^{(d)}(\mathbb{F}_\ell)$ *that are sums of images of rational cusps on* $X_1(p)$*, if for all partitions* $d = n_1 + \ldots + n_m$ *with* $n_1 \geq \cdots \geq n_m$ *and all* m*-tuples* $(d_1 = 1, d_2, \ldots, d_m)$ *of integers representing pairwise distinct elements of* H*, the* d *Hecke operators*

$$(T_i \langle d_j \rangle t)_{\substack{j=1,\ldots,m \\ i=1,\ldots,n_j}} \tag{5.1}$$

are $\mathbb{F}_\ell$*-linearly independent in* $\mathbb{T} \otimes \mathbb{F}_\ell$*, where* $\mathbb{T}$ *is considered as a subalgebra of* $\mathrm{End}_{\mathbb{Q}}(J_H)$.

Maarten Derickx

Sheldon Kamienny

William Stein

Michael Stoll

```python
def get_bad_formal_immersion_data(d):
    """
    This is the Oesterlé for type 1 primes with modular symbols main routine.
    The computation is actually a two step rocket. First Proposition 6.8 of
    Derickx-Kamienny-Stein-Stoll is used to replace Parents polynomial of
    degree 6 bound by something reasonable, and then Corollary 6.4 is used
    to go from something reasonable to the exact list.
    """
    assert d > 0

    p_todo = [int(p) for p in prime_range(11)]
    p_done = {}
    q_to_bad_p = {}

    M = get_M(d)[0]

    for p in prime_range(11, 2 * M * d):
        # first do a relatively cheap test
        if is_formall_immersion_fast(d, p):
            continue
        # this is more expensive
        is_formall = is_formall_immersion(d, p)
        if is_formall:
            if is_formall > 1:
                p_done[int(p)] = is_formall
        else:
            p_todo.append(int(p))

    for p, q_prod in p_done.items():
        for q in prime_divisors(q_prod):
            q_to_bad_p[int(q)] = int(q_to_bad_p.get(q, 1) * p)

    return p_todo, q_to_bad_p
```

TypeTwoPrimes

Condition CC (Momose + B.-Derickx)

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1. *f is odd;*
2. $q^f < p/4$;
3. $q^{2f} + q^f + 1 \not\equiv 0 \pmod{p}$.

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Proposition (B.-Derickx)

Assume GRH. Let K be a number field of degree d, and E/K an elliptic curve possessing a K-rational p-isogeny, for p a Type 2 prime. Then p satisfies

$$p \leq (8d\log(12p) + 16\log(\Delta_K) + 10d + 6)^4.$$

In particular, there are only finitely many primes p as above.

A cubic example

From Superset to Set

Let's run the algorithm on $\mathbb{Q}(\zeta_7)^+$

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How to determine which of these are actually in $\mathrm{IsogPrimeDeg}(\mathbb{Q}(\zeta_7)^+)$? The main ingredient is

Theorem (Box-Gajović-Goodman, 2021)

*For $N \in \{53, 57, 61, 65, 67, 73\}$, the set of cubic points on $X_0(N)$ is finite and listed in Section 5 of [**?**].*

Josha Box

Stevan Gajović

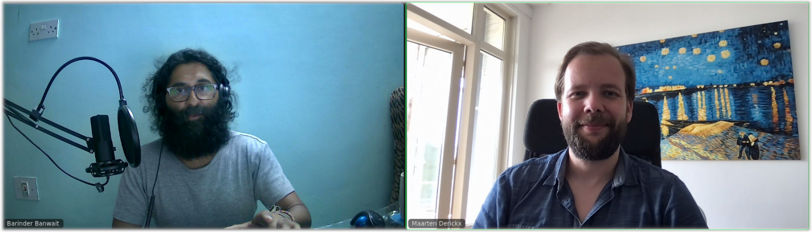
Pip Goodman

The first cubic case of IsogPrimeDeg

Theorem (B.-Derickx)

Assuming GRH,

$$\mathrm{IsogPrimeDeg}(\mathbb{Q}(\zeta_7)^+) = \mathrm{IsogPrimeDeg}(\mathbb{Q})$$



Questions

Question

Can Isogeny Primes v2 be implemented in PARI/GP?

Question

How can checking Type 2 primes be made much faster?