

Elliptic Curves

(PARI-GP version 2.17.1)

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain D **E** = `ellinit(v, {D = 1})`
over \mathbf{Q} $D = 1$
over \mathbf{F}_p $D = p$
over $\mathbf{F}_q, q = p^f$ $D = \text{ffgen}([p, f])$
over \mathbf{Q}_p , precision n $D = O(p^n)$
over \mathbf{C} , current bitprecision $D = 1.0$
over number field K $D = nf$

Points are $[x,y]$, the origin is $[0]$. Struct members accessed as **E.member**:

- All domains: **E.a1,a2,a3,a4,a6, b2,b4,b6,b8, c4,c6, disc, j**
- **E** defined over \mathbf{R} or \mathbf{C}
 x -coords. of points of order 2 **E.roots**
periods / quasi-periods **E.omega, E.eta**
volume of complex lattice **E.area**
- **E** defined over \mathbf{Q}_p
residual characteristic **E.p**
If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**
- **E** defined over \mathbf{F}_q
characteristic **E.p**
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$ **E.no, E.cyc, E.gen**
- **E** defined over \mathbf{Q}
generators of $E(\mathbf{Q})$ (require `elldata`) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from j -invariant `ellfromj(j)`
cubic/quartic/biquadratic to Weierstrass `ellfromeqn(eq)`
add points $P + Q / P - Q$ `elladd(E, P, Q), ellsub`
negate point `ellneg(E, P)`
compute $n \cdot P$ `ellmul(E, P, n)`
sum of Galois conjugates of P `elltrace(E, P)`
check if P is on E `ellisoncurve(E, P)`
order of torsion point P `ellorder(E, P)`
 y -coordinates of point(s) for x `ellordinate(E, x)`
 $[\phi(z), \phi'(z)] \in E(\mathbf{C})$ attached to $z \in \mathbf{C}$ `ellztpoint(E, z)`
 $z \in \mathbf{C}$ such that $P = [\phi(z), \phi'(z)]$ `ellpointtoz(E, P)`
 $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$ to $P \in E(\bar{\mathbf{Q}}_p)$ `ellztopoint(E, z)`
 $P \in E(\bar{\mathbf{Q}}_p)$ to $z \in \bar{\mathbf{Q}}^*/q^{\mathbf{Z}}$ `ellpointtoz(E, P)`
- **Change of Weierstrass models, using** $v = [u, r, s, t]$
change curve E using v `ellchangecurve(E, v)`
change point P using v `ellchangept(E, P, v)`
change point P using inverse of v `ellchangeptinv(P, v)`
is E isomorphic to F ? `ellisom(E, F)`
- **Twists and isogenies**
quadratic twist `elltwist(E, d)`
 n -division polynomial $f_n(x)$ `elldivpol(E, n, {x})`
 $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$; return (ϕ_n, ψ_n^2) `ellxn(E, n, {x})`
isogeny from E to E/G `ellisogeny(E, G)`
apply isogeny to g (point or isogeny) `ellisogenyapply(f, g)`
torsion subgroup with generators `elltors(E)`
- **Formal group**
formal exponential, n terms `ellformalexp(E, {n}, {x})`
formal logarithm, n terms `ellformallog(E, {n}, {x})`
 $\log_E(-x(P)/y(P)) \in \mathbf{Q}_p; P \in E(\mathbf{Q}_p)$ `ellpadiclog(E, p, n, P)`
 P in the formal group `ellformalpoint(E, {n}, {x})`
 $[w/dt, xw/dt]$ `ellformaldifferential(E, {n}, {x})`

$w = -1/y$ in parameter $-x/y$ `ellformalw(E, {n}, {x})`

Curves over finite fields, Pairings

random point on E `random(E)`
 $\#E(\mathbf{F}_q)$ `ellcard(E)`
 $\#E(\mathbf{F}_q)$ with almost prime order `ellsea(E, {tors})`
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ `ellgroup(E)`
is E supersingular? `ellissupersingular(E)`
random supersingular j -invariant over F_2^2 `ellsupersingularj(p)`
Weil pairing of m -torsion pts P, Q `ellweilpairing(E, P, Q, m)`
Tate pairing of $P, Q; P$ m -torsion `elltatepairing(E, P, Q, m)`
Discrete log, find n s.t. $P = [n]Q$ `elllog(E, P, Q, {ord})`

Curves over \mathbf{Q}

Reduction, minimal model

minimal model of E/\mathbf{Q} `ellminimalmodel(E, {\&v})`
quadratic twist of minimal conductor `ellminimaltwist(E)`
 $[k]P$ with good reduction `ellnonsingularmultiple(E, P)`
 E supersingular at p ? `ellissupersingular(E, p)`
affine points of naive height $\leq h$ `ellratpoints(E, h)`

Complex heights

canonical height of P `ellheight(E, P)`
canonical bilinear form taken at P, Q `ellheight(E, P, Q)`
height regulator matrix for pts in L `ellheightmatrix(E, L)`

p -adic heights

cyclotomic p -adic height of $P \in E(\mathbf{Q})$ `ellpadicheight(E, p, n, P)`
... bilinear form at $P, Q \in E(\mathbf{Q})$ `ellpadicheight(E, p, n, P, Q)`
... matrix at vector for pts in L `ellpadicheightmatrix(E, p, n, L)`
... regulator for canonical height `ellpadicregulator(E, p, n, Q)`
Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ `ellpadicfrobenius(E, p, n)`
slope of unit eigenvector of Frobenius `ellpadics2(E, p, n)`

Isogenous curves

matrix of isogeny degrees for \mathbf{Q} -isog. curves `ellisomat(E)`
tree of prime degree isogenies `ellisotree(E)`
a modular equation of prime degree N `ellmodulareqn(N)`

L -function

p -th coeff a_p of L -function, p prime `ellap(E, p)`
 k -th coeff a_k of L -function `ellak(E, k)`
 $L(E, s)$ (using less memory than `lfun`) `elllseries(E, s)`
 $L^{(r)}(E, 1)$ (using less memory than `lfun`) `elll1(E, r)`
a Heegner point on E of rank 1 `ellheegner(E)`
order of vanishing at 1 `ellanalyticrank(E, {eps})`
root number for $L(E, \cdot)$ at p `ellrootno(E, {p})`
modular parametrization of E `elltaniyama(E)`
degree of modular parametrization `ellmoddegree(E)`
compare with $H^1(X_0(N), \mathbf{Z})$ (for $E' \rightarrow E$) `ellweilcurve(E)`
Manin constant of E `ellmaninconstant(E)`

p -adic L function $L_p^{(r)}(E, d, \chi^s)$ `ellpadicl(E, p, n, {s}, {r}, {d})`
BSD conjecture for $L_p^{(r)}(E_D, \chi^0)$ `ellpadicbsd(E, p, n, {D = 1})`
Iwasawa invariants for $L_p(E_D, \tau^i)$ `ellpadiclambda(E, p, D, i)`

Rational points

attempt to compute $E(\mathbf{Q})$ `ellrank(E, {effort}, {points})`
initialize for later `ellrank` calls, `ellrankinit(E)`
saturate $\langle P_1, \dots, P_n \rangle$ wrt. primes $\leq B$ `ellsaturation(E, P, B)`
2-covers of the curve E `ell2cover(E)`

Elldata package, Cremona's database:

db code "11a1" \leftrightarrow [conductor, class, index] `ellconvertname(s)`
generators of Mordell-Weil group `ellgenerators(E)`
look up E in database `ellidentify(E)`

all curves matching criterion `ellsearch(N)`
loop over curves with cond. from a to b `forell(E, a, b, seq)`

Curves over number field K

coeff a_p of L -function `ellap(E, p)`
Kodaira type of \mathfrak{p} -fiber of E `elllocalred(E, p)`
integral model of E/K `ellintegralmodel(E, {\&v})`
minimal model of E/K `ellminimalmodel(E, {\&v})`
minimal discriminant of E/K `ellminimaldisc(E)`
cond, min mod, Tamagawa num $[N, v, c]$ `ellglobalred(E)`
global Tamagawa number `elltamagawa(E)`
test if E has CM `elliscm(E)`
 $P \in E(K)$ n -divisible? $[n]Q = P$ `ellisdivisible(E, P, n, {\&Q})`

L -function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w, |\Im(s)| < h; D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height h).
vector of first n a_k 's in L -function `ellan(E, n)`
init $L^{(k)}(E, s)$ for $k \leq n$ **L** = `lfuninit(E, D, {n = 0})`
compute $L(E, s)$ (n -th derivative) `lfun(L, s, {n = 0})`
 $L(E, 1, r)/(r! \cdot R \cdot \#Sha)$ assuming BSD `ellbsd(E)`

Other curves of small genus

A hyperelliptic curve C is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
check if $[x, y]$ is on C `hyperellisoncurve(C, [x, y])`
 y -coordinates of point(s) for x `hyperellordinate(C, x)`
discriminant of C `hyperelldisc(C)`
Cremona-Stoll reduction `hyperellred(C)`
apply $m = [e, [a, b; c, d], H]$ to model `hyperellchangecurve(C, m)`
minimal discriminant of integral C `hyperellminimaldisc(C)`
minimal model of integral C `hyperellminimalmodel(C)`
reduction of $y^2 + Qy = P$ (genus 2) `genus2red(C, {p})`
Igusa invariants for C of genus 2 `genus2igusa(C)`
affine rational points of height $\leq h$ `hyperellratpoints(C, h)`
find a rational point on a conic, ${}^tXGX = 0$ `qfsolve(G)`
 $[H, U]$ such that $H = c^tUGU$ has minimat `d` `qfminimize(G)`
quadratic Hilbert symbol (at p) `hilbert(x, y, {p})`
all solutions in \mathbf{Q}^3 of ternary form `qfparam(G, x)`
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius `hyperellcharpoly(Q)`
matrix of Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1$ `hyperellpadicfrobenius`

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or *ell* struct (**E.omega**), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean `agm(x, y)`
elliptic j -function $1/q + 744 + \dots$ `ellj(x)`
Weierstrass $\sigma/\wp/\zeta$ function `ellsigma(w, z), ellwp, ellzeta`
periods/quasi-periods `ellperiods(E, {flag}), elleta(w)`
 $(2i\pi/\omega_2)^k E_k(\tau)$ `elleisnum(w, k, {flag})`
modified Dedekind η func. $\prod(1 - q^n)$ `eta(x, {flag})`
Dedekind sum $s(h, k)$ `sumdedekind(h, k)`
Jacobi sine theta function `theta(q, z)`
 k -th derivative at $z=0$ of $\theta(q, z)$ `thetanullk(q, k)`
Weber's f functions `weber(x, {flag})`
modular pol. of level N `polmodular(N, {inv = j})`
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ `polclass(D, {inv = j})`

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